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Geometric Phase in the Breit-Rabi System

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Abstract By using the Lewis-Riesenfeld invariant theory, the geometric phase in the Breit-Rabi system has been studied. It is found that the geometric phase in the cycle case has nothing to do with the coupling constant between electron and atomic nucleus, and the external time-dependent magnetic field.

Keywords Geometric phase · Breit-Rabi system

1 Introduction

It is known that the concept of geometric phase was first introduced by Pancharatnam [1] in studying the interference of classical light in distinct states of polarization. Berry [2] found the quantal counterpart of Pancharatnam's phase in the case of cyclic adiabatic evolution. The extension to non-adiabatic cyclic evolution was developed by Aharonov and Anandan [3]. Samuel et al. [4] generalized the pure state geometric phase by extending it to non-cyclic evolution and sequential projection measurements. The geometric phase is a consequence of quantum kinematics and is thus independent of the detailed nature of the dynamical origin of the path in state space. Mukunda and Simon [5] gave a kinematic approach by taking the path traversed in state space as the primary concept for the geometric phase. Further generalizations and refinements, by relaxing the conditions of adiabaticity, unitarity, and cyclicity of the evolution, have since been carried out [6]. Recently, the geometric phase of the mixed states has also been studied [7–9].

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As we known that the quantum invariant theory proposed by Lewis and Riesenfeld [10] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized by introducing the concept of basic invariants and used to study the geometric phases [11–14] in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations, but also provides us with new insights in many physical phenomena. The concept of Berry's phase has been developed in some different directions [15–27]. In this letter, by using the Lewis-Riesenfeld invariant theory, we shall study the geometric phase in the Breit-Rabi system.

2 Model

The Hamiltonian in the Breit-Rabi system is described as

$$\hat{H} = \lambda \vec{I} \cdot \vec{\sigma} + x(t)\left(I + \frac{1}{2}\right)\hat{\sigma}_z, \tag{1}$$

where $\vec{\sigma}$ and $\vec{I}$ stand for the spins of electron and atomic nucleus, respectively. λ is the coupling constant between electron and atomic nucleus. $\vec{I}^2 = I(I+1)$. $x(t)$ stands for the external time-dependent magnetic field. Using the large quantum number approximation, i.e., $I_z \sim$ constant, we let $\Omega(t) = \lambda I_z + x(t)(I + \frac{1}{2})$, then (1) can be written as

$$\hat{H} = \Omega(t)\hat{\sigma}_z + \lambda[\hat{\sigma}_+\hat{I}_- + \hat{I}_+\hat{\sigma}_-], \tag{2}$$

where $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$, $[\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$, $[\hat{I}_+, \hat{I}_-] = 2\hat{I}_z$, $[\hat{I}_z, \hat{I}_\pm] = \pm\hat{I}_\pm$. If introducing the supersymmetric generators $\hat{Q}_+ = \hat{\sigma}_+\hat{I}_-$ and $\hat{Q}_- = \hat{I}_+\hat{\sigma}_-$, one has

$$\{\hat{Q}_+, \hat{Q}_-\} = \begin{pmatrix} \hat{I}_-\hat{I}_+ & 0 \\ 0 & \hat{I}_+\hat{I}_- \end{pmatrix} \equiv \hat{\hat{M}}. \tag{3}$$

It is easy to find that $\hat{\hat{M}}$, $\hat{I}_\pm$, $\hat{Q}_\pm$, $\hat{\sigma}_\pm$, and $\hat{\sigma}_z$ are the supersymmetric generators and form supersymmetric Lie algebra, namely

$$[\hat{Q}_+, \hat{Q}_-] = \hat{\hat{M}}\hat{\sigma}_z, \qquad [\hat{\hat{M}}, \hat{Q}_-] = 0, \qquad [\hat{\hat{M}}, \hat{Q}_+] = 0, \tag{4}$$

$$\begin{aligned} &\hat{Q}_-^2 = \hat{Q}_+^2 = 0, \qquad [\hat{\hat{M}}, \hat{\sigma}_z] = [\hat{\hat{M}}, \hat{I}_z] = 0, \\ &\{\hat{Q}_-, \hat{\sigma}_z\} = \{\hat{Q}_+, \hat{\sigma}_z\} = 0, \qquad [\hat{\sigma}_z, \hat{I}_z] = 0, \end{aligned} \tag{5}$$

$$\hat{\sigma}_z(\hat{Q}_+ - \hat{Q}_-) = \hat{Q}_+ + \hat{Q}_-, \qquad (\hat{Q}_+ - \hat{Q}_-)^2 = -\hat{\hat{M}}, \tag{6}$$

$$[\hat{Q}_-, \hat{\sigma}_z] = 2\hat{Q}_-, \qquad [\hat{Q}_+, \hat{\sigma}_z] = -2\hat{Q}_+, \qquad [\hat{Q}_-, \hat{I}_z] = -\hat{Q}_-, \qquad [\hat{Q}_+, \hat{I}_z] = \hat{Q}_+, \tag{7}$$

where {, } stands for the anticommuting bracket. Equation (2) becomes

$$\hat{H} = \Omega(t)\hat{\sigma}_z + \lambda(\hat{Q}_+ + \hat{Q}_-). \tag{8}$$

It is easy to find that

$$\hat{\hat{M}} \frac{1}{\sqrt{2}} \begin{pmatrix} |l, m\rangle \\ |l, m+1\rangle \end{pmatrix} = (l-m)(l+m+1) \frac{1}{\sqrt{2}} \begin{pmatrix} |l, m\rangle \\ |l, m+1\rangle \end{pmatrix}, \tag{9}$$

so we can restrict our study in the sub-Hilbert space of the supersymmetric quasi-algebra $\hat{\bar{M}}$, $\hat{Q}_\pm$, $\hat{\sigma}_\pm$, $\hat{I}_\pm$, $\hat{I}_z$ and $\hat{\sigma}_z$. Below, we replace operator $\hat{\bar{M}}$ with the particular eigenvalue $(l-m)(l+m+1)$.

3 Dynamical and Geometric Phase

For self-consistent, we first illustrate the Lewis-Riesenfeld (L-R) invariant theory [10]. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{\varrho}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial\hat{\varrho}(t)}{\partial t} + [\hat{\varrho}(t), \hat{H}(t)] = 0. \tag{10}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$\hat{\varrho}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \tag{11}$$

where $\frac{\partial\lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial|\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \tag{12}$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of (12) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{\varrho}(t)$ only by a phase factor $\exp[i\delta_n(t)]$, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{13}$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (12). Then the general solution of the Schrödinger equation (12) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{14}$$

where

$$\delta_n(t) = \int_0^t dt'\langle\lambda_n, t'|i\frac{\partial}{\partial t'} - \hat{H}(t')|\lambda_n, t'\rangle, \tag{15}$$

and $C_n = \langle\lambda_n, 0|\psi(0)\rangle_s$.

For the system described by Hamiltonian (8), we can define the following invariant

$$\hat{\varrho}(t) = \alpha(t)\hat{Q}_- + \alpha^*(t)\hat{Q}_+ + \beta(t)\hat{\sigma}_z. \tag{16}$$

Substituting (8) and (16) into (10), one has the auxiliary equations

$$i\dot{\alpha}(t) + 2\alpha(t)\Omega(t) - 2\lambda\beta(t) = 0, \tag{17}$$

$$i\dot{\beta}(t) + \lambda\bar{M}[\alpha^*(t) - \alpha(t)] = 0, \tag{18}$$

where dot denotes the time derivative, and $\bar{M}$ denotes the eigenvalue of operator $\hat{\bar{M}}$.

In order to obtain a time-independent invariant, we can introduce the unitary transformation operator $\hat{V}(t)=\exp[\xi(t)\hat{Q}_{-}-\xi^{*}(t)\hat{Q}_{+}]$. It is easy to find that when satisfying the following relations

$$\sin(2\sqrt{\bar{M}}|\xi(t)|)=\frac{\sqrt{\bar{M}}[\alpha(t)\xi^{*}(t)+\alpha^{*}(t)\xi(t)]}{2|\xi(t)|},\qquad \beta(t)=\cos(2\sqrt{\bar{M}}|\xi(t)|),\tag{19}$$

and

$$\frac{\alpha^{*}(t)}{2}[1+\cos(2\sqrt{\bar{M}}|\xi(t)|)]-\frac{\beta(t)\xi^{*}(t)}{\sqrt{\bar{M}}|\xi(t)|}\sin(2\sqrt{\bar{M}}|\xi(t)|)$$
$$+\frac{\alpha(t)\xi^{*2}(t)}{2|\xi(t)|^{2}}[\cos(2\sqrt{\bar{M}}|\xi(t)|)-1]=0,\tag{20}$$

then a time-independent invariant appears

$$\hat{\varrho}_{V}\equiv\hat{V}^{\dagger}(t)\hat{\varrho}(t)\hat{V}(t)=\hat{\sigma}_{z}.\tag{21}$$

According to (19), we can select

$$\xi(t)=\frac{\theta(t)}{\sqrt{2\bar{M}}}\exp[i\gamma(t)],\qquad \alpha(t)=\frac{\sin\theta(t)}{\sqrt{2\bar{M}}}\exp[i\gamma(t)],\qquad \theta(t)=2\sqrt{\bar{M}}|\xi(t)|.\tag{22}$$

From (22), the invariant $\hat{\varrho}(t)$ in (16) becomes

$$\hat{\varrho}(t)=\frac{\sin\theta(t)}{\sqrt{2\bar{M}}}\{\exp[i\gamma(t)]\hat{Q}_{-}+\exp[-i\gamma(t)]\hat{Q}_{+}\}+\cos\theta(t)\hat{\sigma}_{z}.\tag{23}$$

By using the Baker-Campbell-Hausdorff formula [28]

$$\hat{V}^{\dagger}(t)\frac{\partial\hat{V}(t)}{\partial t}=\frac{\partial\hat{L}}{\partial t}+\frac{1}{2!}\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right]+\frac{1}{3!}\left[\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right],\hat{L}\right]+\frac{1}{4!}\left[\left[\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right],\hat{L}\right],\hat{L}\right]+\cdots,\tag{24}$$

with $\hat{V}(t)=\exp[\hat{L}(t)]$, it is easy to find that when satisfying the following equations

$$\frac{\lambda}{2}\sqrt{\bar{M}}\{e^{-i\gamma(t)}[\cos\theta(t)-1]+1+\cos\theta(t)\}-\beta(t)\Omega(t)\sin\theta(t)$$
$$+\frac{1}{\sqrt{2}}[i\dot{\theta}(t)+\dot{\gamma}(t)\theta(t)]+\dot{\gamma}[\sin\theta(t)-\theta(t)]=0,\tag{25}$$

one has

$$\hat{H}_{V}(t)=\hat{V}^{\dagger}(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^{\dagger}(t)\frac{\partial\hat{V}(t)}{\partial t}$$
$$=[\beta(t)\Omega(t)\cos\theta(t)+\lambda\sqrt{\bar{M}}\cos\gamma(t)\sin\theta(t)]\hat{\sigma}_{z}+\dot{\gamma}(t)[1-\cos\theta(t)]\hat{\sigma}_{z}.\tag{26}$$

For $\sigma_{z}=+1$, one has the particular solution of (12):

$$|\psi_{\sigma_{z}=+1}(t)\rangle=\exp\left\{-i\int_{0}^{t}[\dot{\delta}^{d}_{\sigma_{z}=+1}(t')+\dot{\delta}^{g}_{\sigma_{z}=+1}(t')]dt'\right\}\hat{V}(t)\begin{pmatrix}|l,m\rangle\\0\end{pmatrix},\tag{27}$$

where

$$\dot{\delta}^d_{\sigma_z=+1}(t') = \beta(t)\Omega(t)\cos\theta(t) + \lambda\sqrt{(l+m+1)(l-m)}\cos\gamma(t)\sin\theta(t), \tag{28}$$

$$\dot{\delta}^g_{\sigma_z=+1}(t') = \dot{\gamma}(t')[1-\cos\theta(t')]. \tag{29}$$

For $\sigma_z = -1$, one has

$$|\psi_{\sigma_z=-1}(t)\rangle = \exp\left\{-i\int_0^t [\dot{\delta}^d_{\sigma_z=-1}(t') + \dot{\delta}^g_{\sigma_z=-1}(t')]dt'\right\}\hat{V}(t)\begin{pmatrix} 0 \\ |l, m+1\rangle \end{pmatrix}, \tag{30}$$

where

$$\dot{\delta}^d_{\sigma_z=-1}(t') = -\beta(t)\Omega(t)\cos\theta(t) - \lambda\sqrt{(l+m+1)(l-m)}\cos\gamma(t)\sin\theta(t), \tag{31}$$

$$\dot{\delta}^g_{\sigma_z=-1}(t') = -\dot{\gamma}(t')[1-\cos\theta(t')], \tag{32}$$

From (28)–(29), (31)–(32) we conclude that the dynamical and the geometric phase factors of the system are $\exp[-i\int_0^t \dot{\delta}^d_{\sigma_z}(t')dt']$ and $\exp[-i\int_0^t \dot{\delta}^g_{\sigma_z}(t')dt']$ with $\sigma_z = \pm 1$, respectively. In particular, when we consider a cycle in the parameter space of the invariant $\hat{\varrho}(t)$ and let $\theta(t)$=constant, one has from (29) and (32), respectively

$$\delta^g_{\sigma_z}(T) = \begin{cases} 2\pi(1-\cos\theta), & (\sigma_z = +1), \\ -2\pi(1-\cos\theta), & (\sigma_z = -1). \end{cases} \tag{33}$$

Here $2\pi(1-\cos\theta)$ denotes the solid angle over the parameter space of the invariant $\hat{\varrho}(t)$. It is pointed out that the geometric phases in the cycle case have nothing to do with the coupling constant between electron and atomic nucleus, and the external time-dependent magnetic field.

4 Conclusions

In this letter, by using the Lewis-Riesenfeld invariant theory, we have studied the geometric phase in the Breit-Rabi system. We find that the geometric phase in the cycle case has nothing to do with the coupling constant between electron and atomic nucleus, and the external time-dependent magnetic field.

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